Intermittency and universality in fully developed inviscid and weakly compressible turbulent flows

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We performed high resolution numerical simulations of homogenous and isotropic compressible turbulence, with an average 3D Mach number close to 0.3. We study the statistical properties of intermittency for velocity, density and entropy. For the velocity field, which is the primary quantity that can be compared to the isotropic incompressible case, we find no statistical differences in its behavior in the inertial range due either to the slight compressibility or to the different dissipative mechanism. For the density field, we find evidence of "front-like" structures, although no shocks are produced by the simulation.

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Fully-developed three-dimensional turbulence is characterized by an intermittent energy flux from large to small scales. For incompressible and viscous turbulent flows, dissipation occurs close to the Kolmogorov scale η. According to the Kolmogorov theory, the statistical properties of turbulence are scale-invariant within the inertial range $\eta \ll r \ll L_0$, where L_0 is the scale of energy forcing. Intermittency spoils dimensional scale invariance and is the origin of anomalous scaling [1]. There exist a number of numerical and experimental results aimed at investigating the intermittency in turbulence for incompressible and viscous flows. On the other hand, few investigations have been reported so far for the case of weakly compressible and inviscid turbulence, relevant to many astrophysical and geophysical problems [2, 3]. In this letter we present a high-resolution numerical simulation of homogeneous and isotropic, three-dimensional compressible and inviscid turbulence. Our main purpose is to investigate intermittency in the inertial range and compare our finding against known results for incompressible viscous turbulence. For the case of driven and decaying supersonic turbulence similar problems have been addressed in [4, 5]. For this simulation, we use the FLASH 3 component-based simulation framework. While the FLASH framework was primarily designed to treat compressible, reactive flows found in astrophysical environments [6], it is generally applicable to many other types of fluid phenomena. For this simulation, only the compressible hydrodynamics module based on the higher-order Godunov Piecewise Parabolic Method (PPM) was used [7]. The algorithmic methodology of the FLASH framework and the computer science aspects of this turbulence simulation have been described in further detail elsewhere [8]. The equation of motions solved are the Euler fluid equations with forcing:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\boldsymbol{v}\rho) = 0 \qquad (1)$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla} \cdot (\boldsymbol{v}\boldsymbol{v}\rho) = -\boldsymbol{\nabla} P + \boldsymbol{F} \qquad (2)$$

$$\frac{\partial \rho E}{\partial t} + \boldsymbol{\nabla} \cdot [\boldsymbol{v}(\rho E + P)] = 0 \qquad (3)$$

$$P = (\gamma - 1)\rho U \,, \quad E = U + \frac{1}{2}\rho v^2 \,, \qquad (4)$$

where ρ is the density, $\boldsymbol{v}$ the velocity, P the pressure, E the total energy, U the internal energy and γ is the ratio of the specific heats in the system, equation (4) being the equation of state for our system. The effect of the large scale forcing $\boldsymbol{F}$ in (2) gives rise to a turbulent flow whose energy is transferred from scale L_0 towards small scales [8]. The energy input $\int \boldsymbol{v}\boldsymbol{F}\, d^3x$ produces an increase of the internal energy U, which grows in time. One can easily show that the quantity $P\partial_i v_i$ represents the energy transfer from kinetic to internal energy of the flow, which acts primarily on the smallest scales. The mean sound speed increases slightly in time as well, though the 3-D RMS Mach number is roughly 0.3 (1-D Mach number 0.17) throughout. The numerical simulation was done for isotropic and homogeneous forcing [9] with a state-of-the-art resolution of 1856^3 grid points. The integration in time was done for 3 eddy turnover times after an initial transient evolution beginning from rest. A key feature of the numerical simulation is that while the flow is formally inviscid, some viscosity is introduced as a result of the numerical scheme. The numerical treatment of turbulent flows used in the FLASH simulation is sometimes referred to as an Implicit Large Eddy Simulation

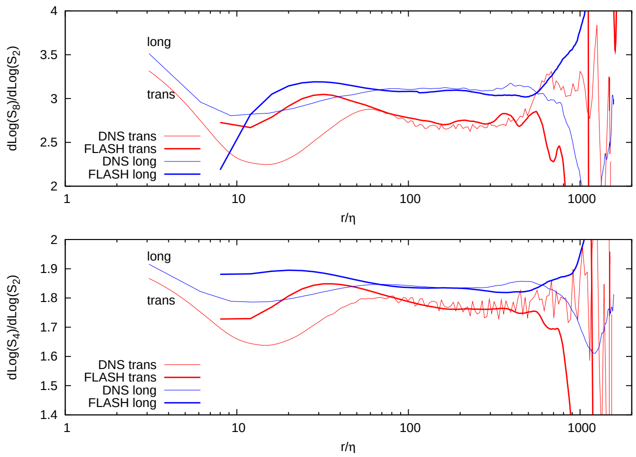

FIG. 1: Local slopes of the 8th-order (top) and 4th-order (bottom) longitudinal (blue lines) and transverse (red lines) structure functions as obtained from the present ILES (bold lines) and compared against a DNS of the Navier-Stokes equations [10] at comparable Reynolds numbers (narrow lines).

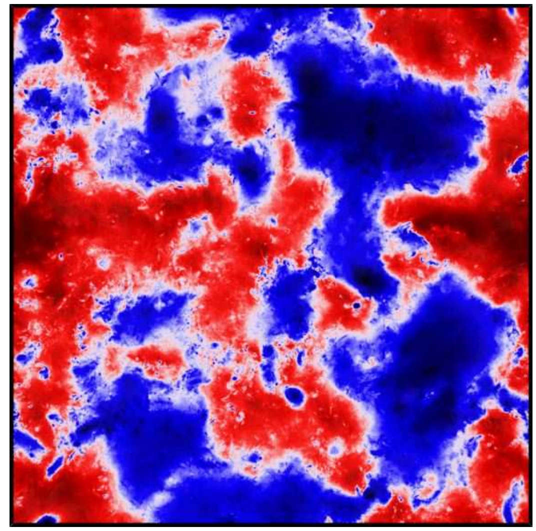

FIG. 2: Two dimensional section of the density field ρ at a given time: large regions with smooth density variations are separated by sharp cliffs.

(ILES), to be distinguished from a full Direct Numerical Simulation (DNS) of the Navier-Stokes equations. Thus, one might expect that the dynamics of the flow may significantly differ from incompressible and viscous high Reynolds number turbulence.

A key question is therefore whether the statistical properties of the inertial range are significantly different from to the homogeneous and isotropic turbulence observed in Navier-Stokes. To answer this question, we employ the concept of anomalous exponents in measuring the statistical properties of velocity fluctuations in the inertial range. In particular, we use the well known method of Extended Self-Similarity (ESS) [11], which allows us to accurately estimate the anomalous exponents of the longitudinal $S_P^{(L)} \equiv \langle [\delta \boldsymbol{v}(\boldsymbol{r}) \cdot \hat{\boldsymbol{r}}]^p \rangle$ (where $\delta \boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{x})$ and $\langle ... \rangle$ means average over the volume and in time) and transverse structure functions $S_P^{(T)} \equiv \langle [\delta \boldsymbol{v}(\boldsymbol{r}_T)]^p \rangle$ (where $\boldsymbol{r}_T \cdot \boldsymbol{v} = 0$). We denote the corresponding scaling exponents by $\zeta_p(L)$ and $\zeta_p(T)$. Our numerical result for $\zeta_p(L), \zeta_p(T)$ agrees remarkably well with previous numerical and experimental data. This is shown in figure 1, which compares the ESS local slope $d\log(S_P)/d\log(S_2)$, for both longitudinal and transverse structure functions $p = 4$ and $p = 8$ with the DNS simulations performed for incompressible Navier-Stokes equation at comparable Reynolds numbers [10]. Figure 1 shows several interesting features. First, there exists a range of scales ($r/\eta \geq 50$) where the local slope is almost constant, i.e. where we can detect accurately an anomalous scaling exponent. Second, as one can see, in the inertial range our ILES results give exactly the same anomalous scaling as that obtained for the incompressible Navier-Stokes equation [10]. There is a clear difference between our results and the Navier-Stokes case in the dissipation range ($r/\eta \leq 20$), where the Navier-Stokes solutions show a well defined "dip" effect (as qualitatively predicted by the multifractal theory [12]), while ILES behaves differently. The different behavior in the dissipation range is expected since the ILES does not dissipate energy in the "standard" Navier-Stokes way. On the other hand, the remarkable agreement in the inertial range allows us to claim that the inertial range properties are independent of the dissipation mechanism. This is an important statement, which has been questioned several times in the past, and supports the conjecture that the statistical properties of turbulence in the inertial range are universal and independent of the dissipation mechanisms. This is one of our main results.

We also note that the difference between longitudinal and transverse scaling exponents observed in homogeneous and isotropic DNS [10], is also seen in our numerical results. This discrepancy is an open theoretical issue, not explainable using standard symmetry argument in homogeneous and isotropic turbulence [13]. Whether this remains true at higher Reynolds numbers is an open question (see also [14] for a discussion on this point).

Although the integration is formally inviscid, there is a net energy transfer from the turbulent kinetic energy $1/2\rho v^2$ to the internal energy. Thus we may consider that an effective viscosity ν_{eff} is acting on the system. In order to estimate ν_{eff} we can proceed as if the Kolmogorov equation –with effective parameters– applies to our case:

$$S_3^{(L)}(r) = -\frac{4}{5}\epsilon_{\text{eff}} r + 6\nu_{\text{eff}} \frac{d}{dr} S_2^{(L)}(r) \qquad (5)$$

A fit of our data with this formula gives, $\epsilon_{\text{eff}} = 0.054, \nu_{\text{eff}} = 8.3 \cdot 10^{-6}$, which corresponds to a Kolmogorov scale, $\eta = (\nu_{\text{eff}}^3/\epsilon_{\text{eff}}^{1/4})$, equivalent to roughly half

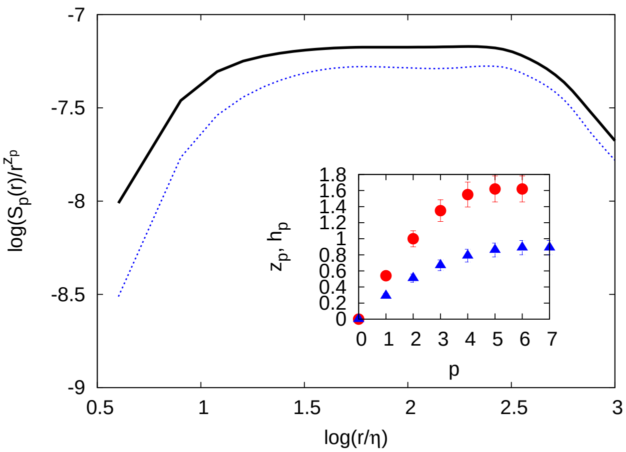

FIG. 3: Compensated structure functions of the density $D_6(r)r^{-z_\infty}$ (solid line) and $D_5(r)r^{-z_\infty}$ (dotted line). In the insert, we show the scaling exponents of the density structure functions (●) and the entropy structure functions (▲). Errors include both statistical fluctuations and the uncertainty in the fit to the inertial range.

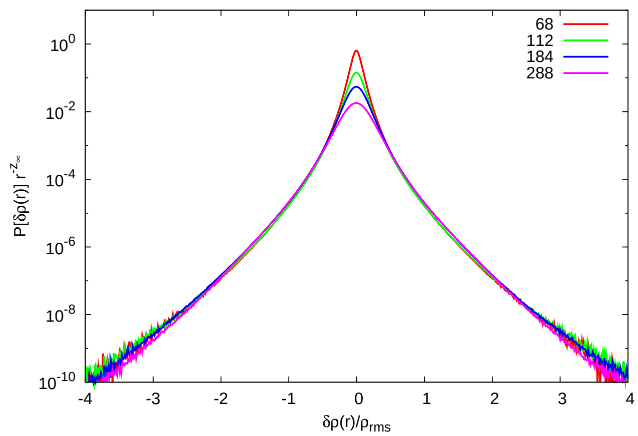

FIG. 4: Rescaled probability distribution function, $P[\delta\rho(r)]r^{-z_\infty}$. Different curves corresponds to different values of r/η in the inertial range (respectively, from top to bottom, 68, 112, 184, 288.)

grid cell and to $R_\lambda \sim 600$. The dynamical effects of the effective viscosity are however different from what one usually observes in the Navier-Stokes equations; i.e., the dissipation range does not behave the same as in the Navier-Stokes solutions. Thus, while we still observe an effective dissipation in the energy flux (i.e., in the third-order structure functions), the behavior in the dissipation range is different because the dissipation mechanism in our simulation is different: the increase of internal energy is due to viscosity resulting from the numerical scheme.

In figure 2 we show the density field in the system at a given time. One can easily recognize the existence of large density gradients due to compressibility. Another interesting quantity to look at is the entropy S defined as $S \equiv \log(P/\rho^\gamma)$. Using equations (1)-(4) one can obtain the following equation for the entropy: $\partial_t S + \boldsymbol{v}\cdot\nabla S = 0$. This equation tells us that S satisfies an equation similar to the case of a passive scalar advected by the velocity vector $\boldsymbol{v}$, although S cannot be considered a passive quantity in this case. Strong variations are also detectable in the field S (not shown). Recently, the statistical properties of density fluctuations have been investigated for supersonic turbulence characterized by large Mach number. A rather large effect is caused by the formation of shock waves and fronts [4, 5, 15]. Here, the 3D Mach number is of order 0.3 on average and, consequently, we should expect that shock waves are not important (although the compressibility degree may reach a maximum excursion where Mach $\sim O(1)$). Actually, it has been shown in experiments and direct numerical simulations of passive scalars, that front-like structures are frequently observed [16, 17]. A front-like structure on a quantity Q is characterized by a "local scaling" property $Q(x+r) - Q(x) \sim const$ for $x+r$ and x selected on the two different sides of the front. If these "front-like" structures play a significant role in the statistical fluctuations of Q, one should observe quite intermittent anomalous scaling for the structure functions of Q. In particular the anomalous exponents should approach a constant value for $p \to \infty$.

In order to study the statistical properties of the density and entropy fluctuations, we can introduce the density structure functions $D_p(r) = \langle[\delta\rho(r)]^p\rangle$ and the entropy structure functions $E_p(r) = \langle[\delta S(r)]^p\rangle$. The ILES result shows that $D_P(r)$ and $E_p(r)$ are scaling functions of r in the inertial range; i.e., $D_p(r) \sim r^{z_p}$ and $E_p(r) \sim r^{h_p}$. The values of z_p and h_p are shown in the insert of figure 3 together with the compensated plot $D_5 r^{-z_5}$ and $D_6 r^{-z_6}$. From figure 3 one can appreciate the quality of the scaling in the inertial range. We remark that the scaling exponents z_p and h_p show quite anomalous behavior. For $p \leq 3$ the values of z_p are larger than the corresponding values of $\zeta_p(L)$ and $\zeta_p(T)$ which means that density is somehow smoother than the velocity field. The anomalous exponents for both the density and the entropy structure functions become constant at large order. In particular, defining the saturation exponents to be z_∞ and h_∞, we estimate $z_\infty = 1.62 \pm 0.10$ and $h_\infty = 1.00 \pm 0.10$. A similar analysis for the pressure field (not shown) shows that the structure functions of the pressure field P have the same scaling exponents and of the same saturation exponents as those for the density field. Using equation (1) and assuming stationarity, one can derive an exact relation for the "energy" flux of density fluctuations $\langle(\delta\rho(r))^2\delta v(r)\rangle$; namely, $\langle(\delta\rho(r))^2\delta v(r)\rangle = 2\langle(\rho(x+r))^2\boldsymbol{\nabla}\cdot\boldsymbol{v}(x)\rangle$. The above equation is fundamental in relating the scaling of the density fluctuations in the inertial range to a flux-like quantity of

"density energy" analogous to the the case of a passive or active scalar. The single point value $\langle(\rho(x))^2\boldsymbol{\nabla}\cdot\boldsymbol{v}(x)\rangle$ should play the same role as "energy dissipation" does in the 4/5 law for the energy cascade in turbulence. This issue deserves more detailed study in the future.

According to the multifractal theory of turbulence, the effect of saturation in the exponents z_p and h_p is equivalent to saying that the tail of the probability distribution $P[\delta\rho(r)]$ should behave as $\sim r^{z_\infty}$ for any r. Thus we should expect that the functions $r^{-z_\infty}P[\delta\rho(r)]$ should collapse on the same "universal" distribution for r in the inertial range [16]. In figure 4 we show that this is exactly the case for ILES result. Let us note that, as shown in figure 3, z_∞ is larger than h_∞. Using the multifractal theory, one can relate the saturation exponent to the fractal dimension of the "front-like" structures; i.e., $D_\rho = 3-z_\infty$ and $D_S = 3-h_\infty$, where D_ρ and D_S are the fractal dimensions of the "front-like" structures for the density and entropy, respectively. Our findings show that D_S is larger than D_ρ; i.e., fronts in the entropy field are easier to form than those in the density field. This result may not be surprising if we observe that large entropy fluctuations are produced by large pressure fluctuations *or* by large density fluctuations. Thus the fractal dimension of the entropy fronts may be larger than those of density and pressure. The above argument implies that entropy is a more intermittent quantity than are density and pressure. The fact that the entropy satisfies a transport equation tells us it is a conserved quantity along Lagrangian trajectories of the numerical simulations. This could suggest some connection between the existence of front-like structures and the behavior of inertial particles in *incompressible* turbulence, where it is known that particles tend to form multifractal sets with correlation dimension as low as 2 [18, 19]. We feel that important results can therefore be achieved by building a systematic "bridge of knowledge" between Lagrangian dynamics in compressible flows and entropy statistics in compressible flows.

Let us summarize our results. Using a numerical simulation of inviscid homogeneous, isotropic weakly-compressible turbulence, we find that the scaling properties of the velocity field in the inertial range are in excellent agreement with those observed in DNS of the Navier-Stokes equations [10]. This result supports the statement that the nature of the dissipation does not affect the statistical properties of the inertial range; i.e., turbulence is universal with respect to the dissipation mechanism. We confirm that transverse and longitudinal structure functions show different scaling properties (up to this Reynolds number). We have also shown that, although almost no shock waves are produced in the simulation, the density fluctuations are characterized by "front-like" structures that determine the tail of the probability distribution of $\delta\rho(r)$. Accordingly, the scaling exponents of density and entropy structure functions, D_p and E_p saturate at large p.

The presence of front-like structures is well known in passive scalars advected by incompressible turbulent flows [16, 17], which suggests that the density field behaves as a passive-like scalar at small Mach numbers. The values of the saturation exponents for the density structure functions are different from those observed in the 3D passive scalar case. This provides further confirmation of the fact that the scaling properties of passive or "passive-like" quantities are not universal with respect to the properties of the advecting velocity field [20] (here the correlation between density and the velocity is certainly different from that observed for true passive scalar fields). The same argument may explain the difference between the scaling properties of the entropy we see in our simulation and those seen at larger Mach numbers in [4, 5].

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